

## A REGULARITY THEOREM FOR HARMONIC MAPS WITH SMALL ENERGY

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### 1. Introduction

This paper studies the regularity problem of harmonic maps in higher dimensions. We consider maps from the unit ball  $B$  in  $\mathbf{R}^n$  ( $n > 2$ ) equipped with a metric  $g$  into a compact submanifold  $N^m$  of  $\mathbf{R}^k$ . We say that  $u \in L_1^2(B, N)$  if  $u \in L_1^2(B, \mathbf{R}^k)$  and  $u(x) \in N$  a.e  $x \in B$ . The energy  $E(u)$  of  $u$  is defined as  $E(u) = \int_B |\nabla u|^2 dv$ . A weakly harmonic map is defined to be the weak solution to the formal Euler-Lagrange equations, which form a nonlinear elliptic system. The equations are

$$(1.1) \quad \Delta u^i(x) = g^{\alpha\beta}(x) A^i \left( \frac{\partial y}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right), \quad i = 1, 2, \dots, K,$$

where  $A_u(X, Y) \in (T_u N)^\perp$  is the second fundamental form of  $N$  given by  $A_u(X, Y) = (D_X Y)^\perp$ .  $X, Y$  are vector fields on  $N$  in a neighborhood of  $u \in N$ .

It is easy to see that  $u$  is harmonic if and only if  $(d/dt)E(u_t)|_{t=0} = 0$ , where  $u_t$  is a 1-parameter family of maps defined by  $u_t(x) = \Pi(u(x) + t\eta(x))$   $\forall \eta \in C_0^\infty(B, \mathbf{R}^k)$ .  $\Pi$  is the nearest point projection of  $\mathbf{R}^k$  into  $N$ .

There is another type of variation that one may consider. One takes  $u_t = u \circ \varphi_t$  for  $\varphi_t$  a 1-parameter family of compactly supported  $C^1$  diffeomorphisms of  $B$  with  $\varphi_0 = \text{Id}$ .  $E(u_t)$  is differentiable in  $t$ . If  $u$  is always critical for this type of variations and if  $u$  is harmonic, then  $u$  is called a stationary map.

So far not much is known about the regularity of weak harmonic maps. For  $n = 2$  it is proved in [6] that a harmonic map with finite energy does not have isolated singularity. A theorem of [7] says that  $u$  has no interior singularity if  $u$  is stationary and  $n = 2$ .

In this paper we generalize the result of [6] to higher dimensions. For  $n > 2$  we cannot expect the finiteness of total energy to be sufficient for the removability of isolated singularity. For example, take any harmonic map  $w$  from  $S^{n-1}$  into  $N$  with finite energy  $E(w)$ . Define map  $u: B \rightarrow N$  by  $u(x) = w(x/|x|)$ .  $u$  is again harmonic with finite energy since  $E(u) = E(w)/(n - 2)$ .  $u$  has singularity at 0 unless  $w$  is constant.

We will assume the smallness of the total energy and show the apparent isolated singularity is removable. Our main result is

**Main Theorem.** *Let  $B$  be the unit ball  $B(0) \subset \mathbb{R}^n$  with a smooth Riemannian metric  $g$ . Let  $u$  be any harmonic map belonging to  $C^\infty(B \setminus \{0\}, N)$ . There exists a constant  $\varepsilon > 0$  independent of  $u$  such that  $u \in C^\infty(B, N)$  provided  $E(u) = \int_B |\nabla u|^2 dv \leq \varepsilon$ .*

Our proof is based on the a priori estimates of  $C^2$  harmonic maps obtained by R. Schoen and K. Uhlenbeck and a monotonicity inequality.

We will present some preliminary results in the next section. In §3 we will prove the theorem assuming that  $u$  is stationary. In §4 we will prove that the monotonicity inequality is true for harmonic maps of finite energy with isolated singularity. This result then enables us to complete the proof of the theorem.

### 2. Preliminary results

**Lemma 1** (*monotonicity inequality*). *Suppose  $u$  is a stationary map from  $B$  into  $N \subset \mathbb{R}^k$ . For  $n > 2$  we have for  $0 < \sigma < \rho < \text{dist}(x_0, \partial B)$*

$$(2.1) \quad \begin{aligned} & e^{C\Delta\rho} \rho^{2-n} \int_{B_\rho(x_0)} |\nabla u|^2 dx - e^{C\Delta\sigma} \sigma^{2-n} \int_{B_\sigma(x_0)} |\nabla u|^2 \\ & \geq 2 \int_{B_\rho(x_0) - B_\sigma(x_0)} e^{C\Delta\sigma} |x - x_0|^{2-n} \left| \frac{\partial u}{\partial r} \right|^2 dx, \end{aligned}$$

where  $\Delta$  and  $C$  are constants,  $B_\rho(x_0)$  (and  $B_\sigma(x_0)$ ) is the geodesic ball of radius  $\rho$  (and  $\sigma$ ) centered at  $x_0$ , respectively. For a proof one can read [5].

**Lemma 2** [8]. *Suppose  $u \in C^2(B_r, N)$  is harmonic with respect to a metric  $g$  on  $B_r$ . Suppose*

$$\Lambda^{-1} j(\delta_{\alpha\beta}) \leq g_{\alpha\beta} \leq \Lambda(\delta_{\alpha\beta}), \quad |\partial_r g_{\alpha\beta}| \leq \Lambda r^{-1}.$$

*There exists  $\varepsilon = \varepsilon(\Lambda, n, N) > 0$  such that if  $r^{2-n} \int_{B_r} |\nabla u|^2 \leq \varepsilon$ , then*

$$(2.2) \quad r^2 \sup_{B_{r/2}} |\nabla u|^2 \leq Cr^{2-n} \int_{B_r} |\nabla u|^2.$$

The proof of Lemma 2 makes use of Lemma 1, noticing that “ $C^2$  harmonic” implies “stationary”. The smallness of the energy is used to ensure that a rescaled version  $v$  of  $u$  satisfying  $e(v)(0) = 1$  and  $\sup e(v) \leq 4$  is defined in a ball  $B_{r_0}$  with  $r_0 \leq 1$ . The boundedness of  $e(v)$  then enables one to use the linear elliptic estimates. Here  $e(v)$  denotes the energy density. For details see [7].

**Lemma 3** (First variation formula). *For a smooth family  $\varphi_t$  of diffeomorphisms which are the identity near  $\partial B$  we let  $u_t = u \circ \varphi_t$ . We then have*

$$(2.3) \quad \left. \frac{d}{dt} E(u_t) \right|_{t=0} = - \int_B \left[ |du|^2 \operatorname{div} X - 2 \langle du(\nabla_{e_i} X), du(e_i) \rangle \right],$$

where  $X =$  variation vector field  $= (d/dt)\varphi_t|_{t=0}$ ,  $e_i, i = 1, 2, \dots, n$ , form an orthonormal basis on  $B$ .

This is a standard result. One can prove it by a change of coordinates. We mention one more result.

**Lemma 4** [4]. *If the image of a harmonic map  $u$  lies in a local strictly convex coordinate chart on  $N$ , then  $u$  is regular.*

### 3. The regularity of stationary maps

In this section we prove the following result:

**Proposition 1.** *If  $u$  is a stationary map from  $B^n$  into  $N \subset \mathbf{R}^k, n > 2$ , with respect to a metric  $g$ , then there exists a constant  $\varepsilon > 0$  such that  $u \in C^\infty(B, N)$  provided that*

$$E(u) \leq \varepsilon \text{ and } u \in C^\infty(B - \{0\}, N).$$

*Proof.* By a change of scale we can reduce to the case that  $g$  is close to the standard metric  $g_0$ . Thus we assume without loss of generality that

$$g_{\alpha\beta} = \delta_{\alpha\beta} + O(\varepsilon), \quad \partial_\nu g_{\alpha\beta} = O(\varepsilon) \text{ in } B.$$

Apply Lemma 2 to the ball  $B_r(x)$  where  $x \neq 0, r = |x|/2$ . We then get

$$r^2 |du|^2(x) \leq Cr^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq CE(u).$$

In the second inequality we have used the monotonicity lemma. Assume  $E(u) \leq \varepsilon$ . We have the estimate

$$(3.1) \quad |du|(x) \leq \frac{C\varepsilon^{1/2}}{|x|}.$$

Take a ball of radius  $r > 0$  centered at 0. Let  $x, y \in \partial B_r(0)$ . Then

$$(3.2) \quad |u(x) - u(y)| \leq \int_T |du| \leq \frac{\varepsilon^{1/2} C}{r} \cdot \Lambda 2\pi r = C\varepsilon^{1/2},$$

where  $\Gamma$  is a geodesic on  $\partial B_r(0)$  connecting  $x$  to  $y$  with length  $\leq \Lambda 2\pi r$ . This shows that  $\text{Osc}(u, \partial B_r) \leq C\epsilon^{1/2}$ .

We want to compare  $u$  with a linear harmonic map  $h: B \rightarrow \mathbf{R}^k$ . To define  $h$ , we take  $0' = \bar{u}(1)$  as the origin of  $\mathbf{R}^k$ , where  $\bar{u}(r) = \int_{\partial B_r} \mu$  is the average of  $u$  over  $\partial B_r$ . Let  $\bar{C}$  be a constant so that  $|A(\nabla u, \nabla u)| \leq \bar{C}|\nabla u|^2$ . Let

$$\lambda = \frac{1}{2} \text{Min}_{Q \in N} \left\{ \text{Max} \left\{ \mu \mid (B_{2\mu}(Q) \cap N) \subset \text{a convex local coordinate chart of } N \right\} \right\},$$

where  $B_{2\mu}(Q)$  is the Euclidean ball of radius  $2\mu$  centered at  $Q$ . Let  $\delta$  be the first point for which  $\bar{u}(r)$  lies on  $\partial B_\lambda(0')$ , i.e.,  $\delta = \max\{r: |\bar{u}(r) - \bar{u}(1)| = \lambda\}$ . We take the first coordinate axis in the direction  $0'\bar{u}(\delta)$ .

Clearly  $\lambda > 0$ . We claim that  $\delta > 0$  if  $0$  is not removable and  $E(u) \leq \epsilon$  for an  $\epsilon > 0$  small. The reason is that  $|u(x) - \bar{u}(|x|)| \leq C \cdot \epsilon^{1/2}$  as a direct consequence of (3.2) and by definition  $\bar{u}(|x|) \in B_\lambda(0') \forall x \in B_1 \setminus B_\delta$ . Thus we can choose  $\epsilon$  small so that  $u(x) \in B_{2\lambda}(\Gamma(0')) \forall x \in B_1 \setminus B_\delta$ . Then  $\delta = 0$  would imply that the image of  $u$  on  $B_1$  lies in a convex local coordinate chart of  $N$ , hence  $u$  would be regular.

Also  $\delta$  is uniformly away from 1 for  $\epsilon$  small as a consequence of the a priori bound of the gradient (3.1). Indeed, we have

$$\lambda = |\bar{u}(\delta) - \bar{u}(1)| \leq \left( \sup_{\delta \leq r \leq 1} |\nabla \bar{u}| \right) (1 - \delta) \leq C\epsilon^{1/2}/\delta,$$

hence  $\delta \leq C\epsilon^{1/2}\lambda^{-1}$ .

Define  $h: B \rightarrow \mathbf{R}^k$  by  $h(x) = (h_1(x), 0, \dots, 0)$ , where

$$h_1(x) = h_1(|x|) = -\frac{\lambda}{\delta^{2-n} - 1} + \frac{\lambda}{\delta^{2-n} - 1} |x|^{2-n}.$$

Note that  $\Delta_{g_0} h = 0$  and  $h(x) = \bar{u}(\delta)$  on  $\partial B_\delta$ ,  $h(x) = \bar{u}(1) = 0'$  on  $\partial B_1$ , where  $\Delta_{g_0}$  is the Euclidean Laplacian.

Observe that for  $n \geq 3$

$$\begin{aligned} \int_{B_1 \setminus B_\delta} \left| \frac{\partial h}{\partial r} \right|^2 r^{2-n} dv &= \int_{B_1 \setminus B_\delta} \left| \frac{\partial h}{\partial r} \right|^2 r^{2-n} \sqrt{g(x)} dx \geq C \int_\delta^1 |h_1'|^2 r dr \\ (3.3) \qquad \qquad \qquad &= C \frac{\lambda^2(n-2)}{2} \frac{\delta^{2-n} + 1}{\delta^{2-n} - 1} \geq \frac{C\lambda^2}{2} > 0. \end{aligned}$$

On the other hand, by Lemma 1 we have

$$(3.4) \qquad \int_{B_1 \setminus B_\delta} \left| \frac{\partial u}{\partial r} \right|^2 r^{2-n} dV \leq C \int_B |\nabla u|^2 dV = CE(u) \leq C\epsilon.$$

Our plan is to show

$$(3.5) \quad \int_{B_1 \setminus B_\delta} \left| \frac{\partial u}{\partial r} - \frac{\partial u}{\partial r} \right|^2 r^{2-n} dV \leq C\epsilon^{1/2},$$

which is a contradiction to (3.3) and (3.4) for  $\epsilon$  small.

Applying Green's formula, we get (denoting  $h_1 = h, u_1 = u$ )

$$\begin{aligned} & \int_{B_1 \setminus B_\delta} r^{2-n} (h - u) \Delta (h - u) \\ &= \int_{B_1 \setminus B_\delta} \nabla [r^{2-n} (h - u)] \cdot \nabla (h - u) \\ & \quad - \int_{\partial B_1 \cup \partial B_\delta} r^{2-n} (h - u) \frac{\partial (h - u)}{\partial r} \\ &= \frac{1}{2} \int_{B_1 \setminus B_\delta} \nabla r^{2-n} \cdot \nabla |h - u|^2 + \int_{B_1 \setminus B_\delta} r^{2-n} |\nabla (h - u)|^2 \\ & \quad - \int_{\partial B_1 \cup \partial B_\delta} r^{2-n} (h - u) \left( \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right) \\ &= \frac{2-n}{2} \int_{\partial B_1 \cup \partial B_\delta} |h - u|^2 r^{2-n} + \int_{B_1 \setminus B_\delta} r^{2-n} |\nabla (h - u)|^2 \\ & \quad - \int_{\partial B_1 \cup \partial B_\delta} r^{2-n} (h - u) \left( \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right) - \frac{1}{2} \int_{B_1 \setminus B_\delta} |h - u|^2 \Delta r^{2-n}. \end{aligned}$$

We get from this

$$\begin{aligned} & \int_{B_1 \setminus B_\delta} r^{2-n} |\nabla (h - u)|^2 \\ &= \frac{2-n}{2} \int_{\partial B_1 \cup \partial B_\delta} |h - u|^2 r^{1-n} + \int_{\partial B_1 \cup \partial B_\delta} r^{2-n} (h - u) \left( \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right) \\ (3.6) \quad & + \int_{B_1 \setminus B_\delta} r^{2-n} (h - u) \Delta u - \int_{B_1 \setminus B_\delta} r^{2-n} (h - u) \Delta h \\ & + \frac{1}{2} \int_{B_1 \setminus B_\delta} |h - u|^2 \Delta r^{2-n}. \end{aligned}$$

Using the fact  $\sup_{\partial B_\delta \cup \partial B_1} |h - u| \leq C\epsilon^{1/2}$ , we can estimate

$$(3.7) \quad \left| \int_{\partial B_\delta} r^{1-n} |h - u|^2 \right| \leq C\epsilon \int_{\partial B_1} \delta^{1-n} \delta^{n-1} \leq C\epsilon,$$

$$\begin{aligned}
 & \int_{\partial B_\delta} r^{2-n} |h - u| \left| \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right| \\
 (3.8) \quad & \leq C\epsilon^{1/2} \int_{\partial B_1} \delta^{2-n} \left[ \lambda(n-2) \frac{\delta^{1-n}}{\delta^{2-n}-1} + \frac{\epsilon^{1/2}}{\delta} \right] \delta^{n-1} \\
 & \leq \lambda C\epsilon^{1/2},
 \end{aligned}$$

where we have used the fact  $\delta$  small to assert

$$\frac{\delta^{1-n}}{\delta^{2-n}-1} = \frac{1}{\delta} \cdot \frac{1}{1-\delta^{2-n}} \leq C\delta^{-1}.$$

Similar estimates can be obtained at  $\partial B_1$ . To deal with the third term in (3.6), observe that

$$\begin{aligned}
 \sup_{B_1 \setminus B_\delta} |h - u| & \leq \sup |h| + \sup |u - \bar{u}| + \sup \bar{u} \\
 & \leq \lambda + C\epsilon^{1/2} + \lambda \leq 2 \cdot \frac{1}{4C} + C\epsilon^{1/2}.
 \end{aligned}$$

Choose  $\epsilon$  small so that

$$(3.9) \quad \bar{C} \sup_{B_1 \setminus B_\delta} |h - u| \leq \frac{1}{2} + \bar{C} < \epsilon^{1/2} \leq 1.$$

We have

$$(3.10) \quad \left| \int_{B_1 \setminus B_\delta} r^{2-n} (h - u) \Delta u \right| \leq \left( \int_{B_1 \setminus B_\delta} r^{2-n} |\nabla u|^2 \right) \bar{C} \sup_{B_1 \setminus B_\delta} |h - u|.$$

The last two terms in (3.6) are bounded by  $C \cdot \epsilon^{1/2}$  as a consequence of our assumptions on  $g_{\alpha\beta}$ .

Since  $h$  is a radial map we can write the left side of (3.6) as

$$\int_{B_1 \setminus B_\delta} r^{2-n} \left| \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right|^2 + \int_{B_1 \setminus B_\delta} r^{2-n} |D_T u|^2,$$

where  $D_T u$  denotes the components tangential to  $\partial B_r$ . Absorbing the tangential term to the left, we get

$$\begin{aligned}
 & \int_{B_1 \setminus B_\delta} r^{2-n} \left| \frac{\partial h}{\partial r} - \frac{\partial u}{\partial r} \right|^2 + \left( 1 - \bar{C} \sup_{B_1 \setminus B_\delta} |h - u| \right) \int_{B_1 \setminus B_\delta} r^{2-n} |D_T u|^2 \\
 (3.11) \quad & \leq C\epsilon^{1/2} + \bar{C} \sup_{B_1 \setminus B_\delta} |h - u| \int_{B_1 \setminus B_\delta} r^{2-n} \left| \frac{\partial u}{\partial r} \right|^2 \\
 & \leq C\epsilon^{1/2} + \epsilon.
 \end{aligned}$$

This inequality completes the proof.

4. Proof of the Main Theorem

We want to show the following extension of Lemma 1.

**Proposition 2.** *If harmonic map  $u$  is  $C^\infty(B \setminus \{0\}, N)$  and if  $E(u) < \infty$ , then we have for  $0 < \rho_1 < \rho_2 \leq 1$*

$$e^{C\Lambda\rho_2}\rho_2^{2-n} \int_{B_{\rho_2}(0)} |\nabla u|^2 - e^{C\Lambda\rho_1}\rho_1^{2-n} \int_{B_{\rho_1}(0)} |\nabla u|^2 \geq \int_{B_{\rho_2}(0) - B_{\rho_1}(0)} e^{C\Lambda r} r^{2-n} \left| \frac{\partial u}{\partial r} \right|^2,$$

where  $C$  and  $\Lambda$  are constants.

*Proof.* Take  $X_\sigma(x) = \psi_\sigma(|x|) \cdot \eta_\tau(|x|) \cdot |x| \cdot (\partial/\partial r)(x)$  for  $\sigma > 0, \tau > 0$  in the first variation formula, where  $\eta_\tau \in C_0^\infty([0, 1], \mathbf{R}^1)$  will be chosen later,  $\psi_\sigma$  is a cut-off function so that  $\psi_\sigma$  is smooth and nonnegative,  $|\psi'_\sigma| \leq 2\sigma^{-1}$  and

$$(4.2) \quad \psi_\sigma(r) \begin{cases} = 0 & \text{if } 0 \leq r \leq \sigma, \\ = 1 & \text{if } r \geq 2\sigma, \\ \leq 1 & \text{elsewhere.} \end{cases}$$

Define  $u_{t,\sigma}: B \rightarrow N$  by

$$(4.3) \quad u_{t,\sigma}(x) = ku(x + tX_\sigma(x)).$$

Note that  $u_{t,\sigma}$  is smooth. Thus we have

$$\frac{d}{dt} E(u_{t,\sigma}) \Big|_{t=0} = 0 \quad \forall \sigma > 0$$

since  $u$  is harmonic.

Let  $\phi \in C^\infty(\mathbf{R}^+, \mathbf{R}^1)$  so that  $\phi(r) = 1$  for  $r \in [0, 1]$ ;  $\phi(r) = 0$  for  $r \in [1 + \sigma_1, \infty)$ ;  $\phi'(r) < 0$  ( $\sigma_1 > 0$  is fixed). Choose  $\eta_\tau(r) = \phi(r/\tau)$  for  $\tau \in [\rho_1, \rho_2]$  in  $X_\sigma$ . Choose an orthonormal basis  $e_1, \dots, e_{n-1}, e_n = \partial/\partial r$ . We have

$$\begin{aligned} \nabla_{e_i} X_\sigma &= \psi_\sigma \eta_\tau \nabla_{re_i} \frac{\partial}{\partial r} \quad \text{for } i = 1, 2, \dots, n-1, \\ \nabla_{\partial/\partial r} X_\sigma &= (\psi_\sigma \eta_\tau)' \frac{\partial}{\partial r}, \end{aligned}$$

where the derivative is taken with respect to  $r$ .

Denote  $x = x(r, \theta)$  and

$$\varepsilon_{ij}(x) = \frac{\partial}{\partial r} \left\langle \left\langle \nabla_{re_i} \left( \frac{\partial}{\partial r} \right), e_j \right\rangle \right\rangle (x), \quad i, j = 1, 2, \dots, n-1.$$

Take a constant  $\Lambda > 0$  so that  $|\varepsilon_{ij}(x)| \leq \Lambda \quad \forall x \in B$ . Then we have for  $i, j = 1, 2, \dots, n-1$

$$\langle \nabla_{e_i} X_\sigma, e_j \rangle = \psi_\sigma \eta_\tau \delta_{ij} + \psi_\sigma \eta_\tau \int_0^r \varepsilon_{ij}(r', \varphi) dr'.$$

Thus

$$\begin{aligned} \operatorname{div} X_\sigma &= \sum_{i=1}^{n-1} \langle \nabla e_i X_\sigma, e_i \rangle + \left\langle \nabla_{\partial/\partial r} X_\sigma, \frac{\partial}{\partial r} \right\rangle \\ &\geq (\psi_\sigma \eta_\tau)' r + n \psi_\sigma \eta_\tau - (n-1) \psi_\sigma \eta_\tau \Lambda r. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\sum_{i=1}^n \langle du(\nabla_{e_i} X_\sigma), du(e_i) \rangle \\ &= \sum_{i=1}^{n-1} \left\langle du\left(\psi_\sigma \eta_\tau \nabla_{re_i} \left(\frac{\partial}{\partial r}\right)\right), du(e_i) \right\rangle \\ &\quad + \left\langle du(\nabla_{\partial/\partial r} X_\sigma), du\left(\frac{\partial}{\partial r}\right) \right\rangle \\ &\leq \sum_{i=1}^{n-1} \psi_\sigma \eta_\tau |du(e_i)|^2 + (n-1) \psi_\sigma \eta_\tau \Lambda r |du|^2 + (\psi_\sigma \eta_\tau r)' \left| \frac{\partial u}{\partial r} \right|^2. \end{aligned}$$

Thus the first variation formula gives

$$\begin{aligned} (4.4) \quad 0 &\geq \int_B [(\psi_\sigma \eta_\tau)' r + n \psi_\sigma \eta_\tau - (n-1) \psi_\sigma \eta_\tau \Lambda r] |du|^2 dv \\ &\quad - 2 \int_B \psi_\sigma \eta_\tau |du|^2 dv - 2 \int_B (\psi_\sigma \eta_\tau r)' \left| \frac{\partial u}{\partial r} \right|^2 dv \\ &\quad - 2(n-1) \int_B \psi_\sigma \eta_\tau \Lambda r |du|^2 dv. \end{aligned}$$

**Claim.**

$$\int_B |\psi'_\sigma| r |du|^2 \eta_\tau dv \rightarrow 0 \quad \text{as } \sigma \rightarrow 0.$$

To see this, use the estimate  $|du|^2(x) \leq C_1 E(u) \sigma^{-2}$  for  $x \in B_{2\sigma} \setminus B_\sigma$ . Since  $|\psi'_\sigma|(x) = 0$  for  $x \in B_\sigma$  or  $x \in B \setminus B_{2\sigma}$  and  $|\psi'_\sigma| \leq 2\sigma^{-1}$ , we get

$$\int_B |\psi'_\sigma| r |du|^2 \eta_\tau dv \leq C_2 E(u) \sigma^n \sigma^{-1} \sigma^{-2} \sigma = C_2 E(u) \sigma^{2-n}.$$

Since  $n \geq 3$ , the conclusion follows. Similarly one can show that  $\lim_{\sigma \rightarrow 0} \int_B \psi_\sigma \eta_\tau |du|^2 = \int_B \eta_\tau |du|^2$ , etc. Letting  $\sigma \rightarrow 0$ , we get from (4.4)

$$\begin{aligned} 0 &\geq \int_B (\eta'_r r + n \eta_\tau - (n-1) \eta_\tau \Lambda r) |du|^2 - 2 \int_B \eta |du|^2 \\ &\quad - 2 \int_B \eta' r \left| \frac{\partial u}{\partial r} \right|^2 - 2(n-1) \int_B \eta_\tau \Lambda r |du|^2. \end{aligned}$$



It follows that

$$2\tau \frac{\partial}{\partial \tau} \int_B \eta \left| \frac{\partial u}{\partial r} \right|^2 \leq 3(n-1) \int_B \eta r \Lambda |du|^2 + \tau \frac{\partial}{\partial \tau} \int_B \eta_\tau |du|^2 + (2-n) \int_B \eta |du|^2$$

$$\leq 3(n-1)(1 + \sigma_1) \int_B \eta \Lambda |du|^2 + \tau \frac{\partial}{\partial \tau} \int_B \eta_\tau |du|^2 + (2-n) \int_B \eta |du|^2.$$

Multiplying by  $\tau^{1-n} e^{C\Lambda\tau}$  for  $C = 3(n-1)$ , we have

$$e^{C\Lambda\tau} 2\tau^{2-n} \frac{\partial}{\partial \tau} \int_B \eta_\tau \left| \frac{\partial u}{\partial r} \right|^2 \leq \frac{\partial}{\partial \tau} \left( e^{C\Lambda\tau} \tau^{2-n} \int_B \eta_\tau |du|^2 \right) + \sigma_1 e^{C\Lambda\tau} C \tau^{2-n} \int_B \eta_\tau |du|^2.$$

Integrate over  $[\rho_1, \rho_2]$  and let  $\sigma_1 \rightarrow 0$ . We then get

$$e^{C\Lambda\rho_1} \rho_1^{2-n} \int_{B_{\rho_1}} |du|^2 - e^{C\Lambda\rho_2} \rho_2^{2-n} \int_{B_{\rho_2}} |du|^2 \geq 2 \int_{B_{\rho_2} \setminus B_{\rho_1}} e^{C\Lambda r} r^{2-n} \left| \frac{\partial u}{\partial r} \right|^2.$$

In the above computation we denote  $B = B_1(x_0)$  and  $B_{\rho_2} = B_{\rho_2}(x_0)$ ,  $B_{\rho_1} = B_{\rho_1}(x_0)$ .

*Proof of the Main Theorem.* Under the assumption of the theorem we have for  $x \in B_1 \setminus \{0\}$ ,  $|x| < 1/2$ ,

$$\frac{|x|^{2-n}}{2} \int_{B_{|x|/2}(x)} |\nabla u|^2 \leq C(2|x|)^{2-n} \int_{B_{2|x|}(0)} |\nabla u|^2 \leq CE(u).$$

So the estimate (3.1) still holds. Here we have applied Proposition 2 with  $\rho_1 = 2|x|$ ,  $\rho_2 = 1$ . Then the same argument in the proof of Proposition 1 can be carried through.

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